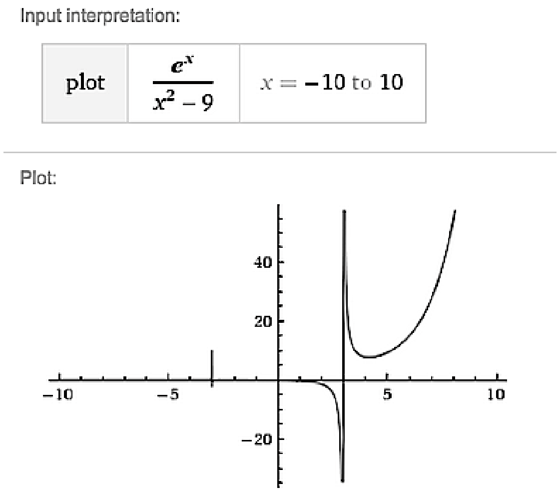
Objectives:

- Use calculus to ensure we have accurate graphs when we use computers for assistance.

Example: Consider the function $f(x) = \dfrac{e^x}{x^2-9}$. We want to produce a graph of f that shows all interesting characteristics of f. So we want to capture all intervals of increase and decrease, extreme values, intervals of concavity, and inflection points.

First, let's try graphing f online with WolframAlpha:



This doesn't seem very useful... There is an asymptote drawn like a regular function and it doesn't seem like the negative values of x are in the domain at all!

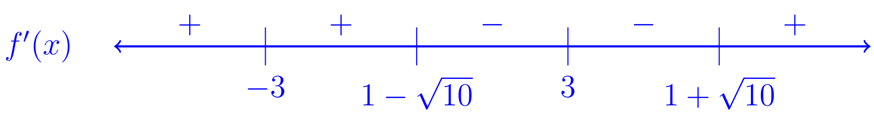
Let's use calculus to do better: The first derivative will tell us about intervals of increase and decrease so let's look there first.

$$f'(x) = \frac{e^x(x^2-9)+e^x(2x)}{(x^2-9)^2} = \frac{e^x}{(x^2-9)^2}(x^2-9-2x)$$

Since $\dfrac{e^x}{x^2-9}$ is always positive, the sign of f' depends only on the sign of x^2-9-2x. Using the quadratic formula to find the zeros of this quadratic, we have

$$x = \frac{2\pm\sqrt{4+36}}{2} = \frac{2\pm\sqrt{40}}{2} = \frac{2\pm2\sqrt{10}}{2} = 1\pm\sqrt{10}$$

Note that $\sqrt{10}$ is between 3 and 4. Now we need to find the rest of the critical points of f: where f is undefined. This occurs when $x^2-9=0$ so $x=-3$, and $x=-3$ are also critical points.
Now we can build a sign chart:



So $f(x)$ is increasing from $(-\infty,-3)$ and $(1+\sqrt{10},\infty)$; $f(x)$ is decreasing from $(-3,3)$ and $(3,1+\sqrt{10})$.

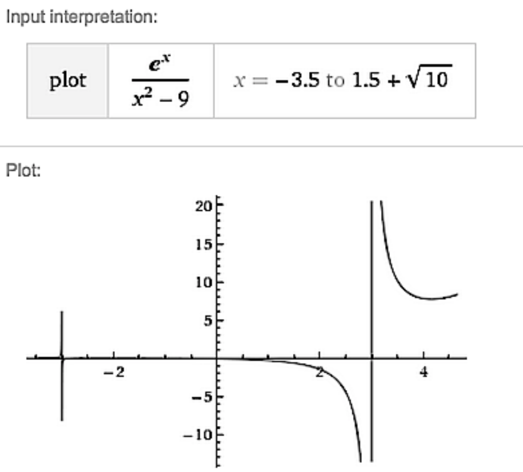
Using the first derivative test, f has a local minimum at $x = 1+\sqrt{10}$ and a local maximum at $x = 1-\sqrt{10}$.

Reality check: What have we done? We found that in order for our graph to show intervals of increase and decrease, and local extrema, we need to have our domain include $(-3.5, 1.5+\sqrt{10})$ (ish).

Now what? To find intervals of concavity and inflection points, we need the second derivative.
Quotient rule magic:

$$f''(x) = \frac{e^x(x^4 - 4x^3 - 12x^2 + 36x + 99)}{(x^2 - 9)^3}$$

This has no real roots so there are no inflection points and the only possible concavity changes occur where $f''(x)$ is undefined ($x = \pm 3$) but we already know that these need to be included in our frame for making the graph of f accurate.
Last reality check: Now we know that in order to capture all interesting information of the graph of f, we should choose our domain to be an interval like $(-3.5, 1.5 + \sqrt{10})$ (ish).



So this is better but not great. It is still hard to see what's going on on the negative axis but we could make multiple graphs to get a better idea: